POD-RBF

Let the response data be stored in a matrix called the snapshot U:

$$\mathbf{U} = \begin{bmatrix} u_1^1 & u_1^2 & \cdots & u_1^S \\ u_2^1 & u_2^2 & & u_2^S \\ \vdots & \vdots & \ddots & \vdots \\ u_N^1 & u_N^2 & \cdots & u_N^S \end{bmatrix} \tag{1}$$

Where N is the total number of sensor points. And S is the number of snapshot vectors $\mathbf{u}_i$, each representing the response corresponding a unique set of parameters. On the other hand, the matrix $\mathbf{P}$ stores parameters.

The main purpose of POD is to extract a set of orthogonal vectors $\mathbf{\Phi}$, called POD basis vectors. Where $\mathbf{\Phi}$ is related to the matrix $\mathbf{U}$ by the following linear relationship:

$$\mathbf{U} = \mathbf{\Phi} \cdot \mathbf{A} \tag{2}$$

Thus, according to the orthogonally of $\mathbf{\Phi}$, the amplitude matrix A can be computed from:

$$\mathbf{A} = \mathbf{\Phi}^T \cdot \mathbf{U} \tag{3}$$

These vectors can also be calculated through the POD operation [1, 2], by extracting the eigenvectors of the covariance matrix $\mathbf{C}$

$$\mathbf{C} = \mathbf{U}^T \cdot \mathbf{U} \tag{4}$$

$$\mathbf{\Phi} = \mathbf{U} \cdot \mathbf{V} \cdot \mathbf{\Lambda}^{-1/2}, \tag{5}$$

Where $\mathbf{V}$ is the matrix storing the normalized eigenvectors of the covariance matrix $\mathbf{C}$, and $\mathbf{\Lambda}$ the diagonal matrix storing its eigenvalues. This method gives eigenvalues that are stored in a descending order. Also called POD directions. And they represent the original data.

Because lower values POD directions hold very little representation, they can be sacrificed in order to reduce the size of the model. With negligible influence the accuracy of the representation. This step is called the truncation. Consequently, the amplitude matrix $\widehat{\mathbf{A}}$ is given by:

$$\widehat{\mathbf{A}} = \widehat{\mathbf{\Phi}}^T \cdot \mathbf{U} \tag{6}$$

Where $\widehat{\mathbf{\Phi}}$ is the sub-space extracted out from $\mathbf{\Phi}$ by preserving the first K (K $\ll$ S) columns of $\mathbf{\Phi}$ that correspond to the largest eigenvalues. We can then reconstruct the original data using the reduced model.

$$\mathbf{U} = \widehat{\mathbf{\Phi}} \cdot \widehat{\mathbf{A}} \tag{7}$$

On the other hand, Radial Basis Function (RBF) allows the generation of the responses corresponding to new sets parameters by interpolation between values of the initial selection. This is done using the matrix $\mathbf{B}$ containing the coefficients calculated by the following equation:

$$\mathbf{B} = \mathbf{A} \cdot \mathbf{G}^{-1} \tag{8}$$

Where $\mathbf{A}$ is already calculated by equation (3) and **G** is defined as the matrix of interpolation functions $\mathrm{g_i}(\mathrm{p})$. In this work, the identity RBF been employed [3-5]:

$$\mathbf{g}_\mathrm{i} = |\mathbf{P} - \mathbf{P}_\mathrm{i}| \tag{9}$$

In Equation (9), the set of interpolation functions $\mathbf{g}_\mathrm{i}(\mathbf{P})$ is arbitrary chosen from the matrix **G** and $\mathbf{P}_\mathrm{i}$ is the parameter corresponding to $\mathbf{U}_\mathrm{i}$ (i=1,2,…,S). The argument of the i-th RBF is the distance$|\mathbf{P} - \mathbf{P}_\mathrm{i}|$, $\mathbf{P}$ and $\mathbf{P}_\mathrm{i}$ being respectively current and reference parameters.

After the evaluation of the coefficient matrix $\mathbf{B}$, a low-dimensional model of (8) can be put in vector form, as follows:

$$\mathbf{a}(\mathbf{P}) = \mathbf{B} \cdot \mathbf{g}(\mathbf{P}) \tag{10}$$

By defining the amplitude vector $\mathbf{a}(\mathbf{P})$ as a function of parameters **P**, the Equation (7) can be rewritten as the approximation of the snapshot $\mathbf{u}$ corresponding to a new parameter vector P:

$$\mathbf{u}(\mathbf{P}) = \widehat{\mathbf{\Phi}} \cdot \mathbf{a}(\mathbf{P}) \tag{11}$$

The reduced model given by Equation (11) is defined as the POD-RBF network. It is able to reproduce unknown response fields corresponding to any set of parameter $\mathbf{P}$ However, it should be noted that this model can lead to results with weak precision when making extrapolation outside the range of known parameters **P**.

References

[1] V. Buljak and G. Maier, "Proper orthogonal decomposition and radial basis functions in material characterization based on instrumented indentation," *Engineering Structures*, vol. 33, pp. 492-501, 2011.

[2] B. Benaissa, N. A. Hocine, I. Belaidi, A. Hamrani, and V. Pettarin, "Crack identification using model reduction based on proper orthogonal decomposition coupled with radial basis functions," *Structural and Multidisciplinary Optimization*, pp. 1-10, 2016.

[3] B. Benaissa, N. A. Hocine, S. Khatir, M. K. Riahi, and S. Mirjalili, "YUKI algorithm and POD-RBF for Elastostatic and dynamic crack identification," *Journal of Computational Science*, p. 101451, 2021.

[4] B. Benaissa, F. Hendrichovsky, K. Yishida, M. Koppen, and P. Sincak, "Phone application for indoor localization based on Ble signal fingerprint," in *2018 9th IFIP International Conference on New Technologies, Mobility and Security (NTMS)*, 2018, pp. 1-5.

[5] B. Benaissa, K. Yoshida, M. Köppen, and F. Hendrichovsky, "Updatable indoor localization based on BLE signal fingerprint," in *2018 International Conference on Applied Smart Systems (ICASS)*, 2018, pp. 1-6.